Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2015

Mathematics

Higher level

Paper 1

27 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \big(2\cos(5x-3)\big)5 \quad \big(=10\cos(5x-3)\big) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\big(2\cos(5x-3)\big)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = 0.25 + 0.6 - 0.7$ ***M1***

$= 0.15$ ***A1***

[2 marks]

(b) **EITHER**

$\mathrm{P}(A)\,\mathrm{P}(B)\left(= 0.25 \times 0.6\right) = 0.15$ ***A1***

$= \mathrm{P}(A \cap B)$ so independent ***R1***

OR

$P\left(A|B\right) = \dfrac{P\left(A \cap B\right)}{P\left(B\right)} = \dfrac{0.15}{0.6} = 0.25$ ***A1***

$= P\left(A\right)$ so independent ***R1***

Note: Allow follow through for incorrect answer to (a) that will result in events being dependent in (b).

[2 marks]

Total [4 marks]

2. $(3-x)^4 = 1.3^4 + 4.3^3(-x) + 6.3^2(-x)^2 + 4.3(-x)^3 + 1(-x)^4$ or equivalent ***(M1)(A1)***

$= 81 - 108x + 54x^2 - 12x^3 + x^4$ ***A1A1***

Note: ***A1*** for ascending powers, ***A1*** for correct coefficients including signs.

[4 marks]

3. $\tan x + \tan 2x = 0$

$\tan x + \dfrac{2\tan x}{1-\tan^2 x} = 0$ ***M1***

$\tan x - \tan^3 x + 2\tan x = 0$ ***A1***

$\tan x\left(3 - \tan^2 x\right) = 0$ ***(M1)***

$\tan x = 0 \Rightarrow x = 0,\ x = 180^\circ$ ***A1***

Note: If $x = 360^\circ$ seen anywhere award ***A0***

$\tan x = \sqrt{3} \Rightarrow x = 60^\circ,\ 240^\circ$ ***A1***

$\tan x = -\sqrt{3} \Rightarrow x = 120^\circ,\ 300^\circ$ ***A1***

[6 marks]

4. (a) attempt to differentiate $f(x) = x^3 - 3x^2 + 4$ ***M1***

$f'(x) = 3x^2 - 6x$ ***A1***

$= 3x(x-2)$

(Critical values occur at) $x = 0$, $x = 2$ ***(A1)***

so f decreasing on $x \in \left]0, 2\right[$ (or $0 < x < 2$) ***A1***

[4 marks]

(b) $f''(x) = 6x - 6$ ***(A1)***

setting $f''(x) = 0$ ***M1***

$\Rightarrow x = 1$

coordinate is $(1, 2)$ ***A1***

[3 marks]

Total [7 marks]

5. any attempt at integration by parts ***M1***

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$ ***(A1)***

$\frac{dv}{dx} = x^3 \Rightarrow v = \frac{x^4}{4}$ ***(A1)***

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \int_1^2 \frac{x^3}{4}dx$ ***A1***

Note: Condone absence of limits at this stage.

$= \left[\frac{x^4}{4}\ln x\right]_1^2 - \left[\frac{x^4}{16}\right]_1^2$ ***A1***

Note: Condone absence of limits at this stage.

$= 4\ln 2 - \left(1 - \frac{1}{16}\right)$ ***A1***

$= 4\ln 2 - \frac{15}{16}$ ***AG***

[6 marks]

6. (a) any attempt to use sine rule ***M1***

$$\frac{\text{AB}}{\sin\frac{\pi}{3}}=\frac{\sqrt{3}}{\sin\left(\frac{2\pi}{3}-\theta\right)}$$ ***A1***

$$=\frac{\sqrt{3}}{\sin\frac{2\pi}{3}\cos\theta-\cos\frac{2\pi}{3}\sin\theta}$$ ***A1***

Note: Condone use of degrees.

$$=\frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta}$$ ***A1***

$$\frac{\text{AB}}{\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta}$$

$$\therefore \text{AB}=\frac{3}{\sqrt{3}\cos\theta+\sin\theta}$$ ***AG***

[4 marks]

(b) **METHOD 1**

$$(\text{AB})'=\frac{-3\left(-\sqrt{3}\sin\theta+\cos\theta\right)}{\left(\sqrt{3}\cos\theta+\sin\theta\right)^2}$$ ***M1A1***

setting $(\text{AB})'=0$ ***M1***

$$\tan\theta=\frac{1}{\sqrt{3}}$$

$$\theta=\frac{\pi}{6}$$ ***A1***

continued…

Question 6 continued

METHOD 2

$$\mathrm{AB} = \frac{\sqrt{3}\sin\frac{\pi}{3}}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

AB minimum when $\sin\left(\frac{2\pi}{3} - \theta\right)$ is maximum ***M1***

$\sin\left(\frac{2\pi}{3} - \theta\right) = 1$ ***(A1)***

$\frac{2\pi}{3} - \theta = \frac{\pi}{2}$ ***M1***

$\theta = \frac{\pi}{6}$ ***A1***

METHOD 3

shortest distance from B to AC is perpendicular to AC ***R1***

$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ ***M1A2***

[4 marks]

Total [8 marks]

7. (a) **METHOD 1**

$$z^3 = -\frac{27}{8} = \frac{27}{8}(\cos\pi + \mathrm{i}\sin\pi)$$ ***M1(A1)***

$$= \frac{27}{8}\left(\cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi)\right)$$ ***(A1)***

$$z = \frac{3}{2}\left(\cos\left(\frac{\pi + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2n\pi}{3}\right)\right)$$ ***M1***

$$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$$

$$z_2 = \frac{3}{2}\left(\cos\pi + \mathrm{i}\sin\pi\right),$$

$$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

continued…

Question 7 continued

METHOD 2

$8z^3 + 27 = 0$

$\Rightarrow z = -\frac{3}{2}$ so $(2z+3)$ is a factor

Attempt to use long division or factor theorem: ***M1***

$\Rightarrow 8z^3 + 27 \equiv (2z+3)(4z^2 - 6z + 9)$

$\Rightarrow 4z^2 - 6z + 9 = 0$ ***A1***

Attempt to solve quadratic: ***M1***

$z = \frac{3 \pm 3\sqrt{3}\,\mathrm{i}}{4}$ ***A1***

$z_1 = \frac{3}{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$z_2 = \frac{3}{2}(\cos\pi + \mathrm{i}\sin\pi)$,

$z_3 = \frac{3}{2}\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right)$. ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $(re^{\mathrm{i}\theta})$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

continued…

Question 7 continued

METHOD 3

$8z^3+27=0$

Substitute $z=x+\mathrm{i}\,y$ ***M1***

$8\left(x^3+3\mathrm{i}\,x^2y-3xy^2-\mathrm{i}\,y^3\right)+27=0$

$\Rightarrow 8x^3-24xy^2+27=0$ and $24x^2y-8y^3=0$ ***A1***

Attempt to solve simultaneously: ***M1***

$8y\left(3x^2-y^2\right)=0$

$y=0, y=x\sqrt{3}, y=-x\sqrt{3}$

$\Rightarrow\left(x=-\frac{3}{2}, y=0\right), x=\frac{3}{4}, y=\pm\frac{3\sqrt{3}}{4}$ ***A1***

$z_1=\frac{3}{2}\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right),$

$z_2=\frac{3}{2}\left(\cos\pi+\mathrm{i}\sin\pi\right),$

$z_3=\frac{3}{2}\left(\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}\right).$ ***A2***

Note: Accept $-\frac{\pi}{3}$ as the argument for z_3.

Note: Award ***A1*** for 2 correct roots.

Note: Allow solutions expressed in Eulerian $\left(re^{\mathrm{i}\theta}\right)$ form.

Note: Allow use of degrees in mod-arg (r-cis) form only.

[6 marks]

continued…

Question 7 continued

(b) **EITHER**

Valid attempt to use area $= 3\left(\frac{1}{2}ab\sin C\right)$ ***M1***

$= 3 \times \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{\sqrt{3}}{2}$ ***A1A1***

Note: Award ***A1*** for correct sides, ***A1*** for correct $\sin C$.

OR

Valid attempt to use area $= \frac{1}{2}$ base $\times$ height ***M1***

area $= \frac{1}{2} \times \left(\frac{3}{4} + \frac{3}{2}\right) \times \frac{6\sqrt{3}}{4}$ ***A1A1***

Note: ***A1*** for correct height, ***A1*** for correct base.

THEN

$= \frac{27\sqrt{3}}{16}$ ***AG***

[3 marks]

Total [9 marks]

8. EITHER

$x = \arctan t$ ***(M1)***

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+t^2}$$ ***A1***

OR

$t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$ ***(M1)***

$= 1 + \tan^2 x$ ***A1***

$= 1 + t^2$

THEN

$$\sin x = \frac{t}{\sqrt{1+t^2}}$$ ***(A1)***

Note: This ***A1*** is independent of the first two marks

$$\int \frac{\mathrm{d}x}{1+\sin^2 x} = \int \frac{\frac{\mathrm{d}t}{1+t^2}}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2}$$ ***M1A1***

Note: Award ***M1*** for attempting to obtain integral in terms of t and $\mathrm{d}t$

$$= \int \frac{\mathrm{d}t}{\left(1+t^2\right)+t^2} = \int \frac{\mathrm{d}t}{1+2t^2}$$ ***A1***

$$= \frac{1}{2}\int \frac{\mathrm{d}t}{\frac{1}{2}+t^2} = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \arctan\left(\frac{t}{\frac{1}{\sqrt{2}}}\right)$$ ***A1***

$$= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan x\right)(+c)$$ ***A1***

[8 marks]

9. (a) $a > 0$ ***A1***

$a \neq 1$ ***A1***

[2 marks]

(b) **METHOD 1**

$\log_x y = \dfrac{\ln y}{\ln x}$ and $\log_y x = \dfrac{\ln x}{\ln y}$ ***M1A1***

Note: Use of any base is permissible here, not just "e".

$\left(\dfrac{\ln y}{\ln x}\right)^2 = 4$ ***A1***

$\ln y = \pm 2 \ln x$ ***A1***

$y = x^2$ or $\dfrac{1}{x^2}$ ***A1A1***

METHOD 2

$\log_y x = \dfrac{\log_x x}{\log_x y} = \dfrac{1}{\log_x y}$ ***M1A1***

$(\log_x y)^2 = 4$ ***A1***

$\log_x y = \pm 2$ ***A1***

$y = x^2$ or $y = \dfrac{1}{x^2}$ ***A1A1***

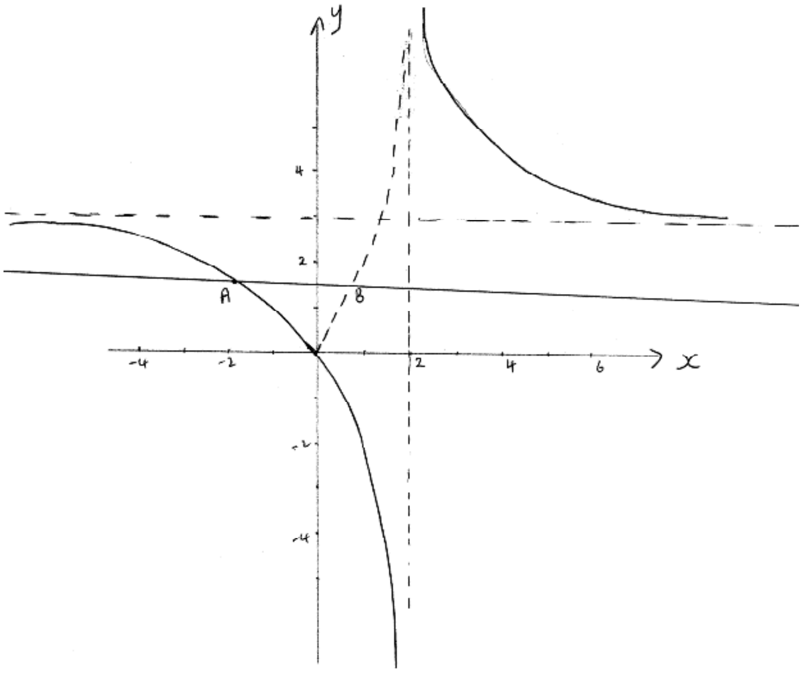
Note: The final two ***A*** marks are independent of the one coming before.

[6 marks]

Total [8 marks]

Section B

10. (a)



Note: In the diagram, points marked A and B refer to part (d) and do not need to be seen in part (a).

shape of curve *A1*

Note: This mark can only be awarded if there appear to be both horizontal and vertical asymptotes.

intersection at $(0, 0)$ *A1*
horizontal asymptote at $y = 3$ *A1*
vertical asymptote at $x = 2$ *A1*

[4 marks]

(b) $y = \dfrac{3x}{x-2}$

$xy - 2y = 3x$ *M1A1*
$xy - 3x = 2y$
$x = \dfrac{2y}{y-3}$
$\left(f^{-1}(x)\right) = \dfrac{2x}{x-3}$ *M1A1*

Note: Final M1 is for interchanging of x and y, which may be seen at any stage.

[4 marks]

continued…

Question 10 continued

(c) **METHOD 1**

attempt to solve $\dfrac{2x}{x-3}=\dfrac{3x}{x-2}$ ***(M1)***

$2x(x-2)=3x(x-3)$

$x\left[2(x-2)-3(x-3)\right]=0$

$x(5-x)=0$

$x=0$ or $x=5$ ***A1A1***

METHOD 2

$x=\dfrac{3x}{x-2}$ or $x=\dfrac{2x}{x-3}$ ***(M1)***

$x=0$ or $x=5$ ***A1A1***

[3 marks]

(d) **METHOD 1**

at $\mathrm{A}:\dfrac{3x}{x-2}=\dfrac{3}{2}$ AND at $\mathrm{B}:\dfrac{3x}{x-2}=-\dfrac{3}{2}$ ***M1***

$6x=3x-6$

$x=-2$ ***A1***

$6x=6-3x$

$x=\dfrac{2}{3}$ ***A1***

solution is $-2<x<\dfrac{2}{3}$ ***A1***

[4 marks]

METHOD 2

$\left(\dfrac{3x}{x-2}\right)^2<\left(\dfrac{3}{2}\right)^2$ ***M1***

$9x^2<\dfrac{9}{4}\left(x-2\right)^2$

$3x^2+4x-4<0$

$\left(3x-2\right)\left(x+2\right)<0$

$x=-2$ ***(A1)***

$x=\dfrac{2}{3}$ ***(A1)***

solution is $-2<x<\dfrac{2}{3}$ ***A1***

[4 marks]

continued…

Question 10 continued

(e) $-2 < x < 2$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

Note: If working is shown, then ***A*** marks may only be awarded following correct working.

[2 marks]

Total [17 marks]

11. (a) $g \circ f(x) = \frac{\tan x + 1}{\tan x - 1}$ ***A1***

$x \neq \frac{\pi}{4},\ 0 \leq x < \frac{\pi}{2}$ ***A1***

[2 marks]

(b) $\frac{\tan x + 1}{\tan x - 1} = \frac{\frac{\sin x}{\cos x} + 1}{\frac{\sin x}{\cos x} - 1}$ ***M1A1***

$= \frac{\sin x + \cos x}{\sin x - \cos x}$ ***AG***

[2 marks]

(c) **METHOD 1**

$\frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ ***M1(A1)***

$\frac{dy}{dx} = \frac{\left(2\sin x \cos x - \cos^2 x - \sin^2 x\right) - \left(2\sin x \cos x + \cos^2 x + \sin^2 x\right)}{\cos^2 x + \sin^2 x - 2\sin x \cos x}$

$= \frac{-2}{1 - \sin 2x}$

Substitute $\frac{\pi}{6}$ into any formula for $\frac{dy}{dx}$ ***M1***

$\frac{-2}{1 - \sin \frac{\pi}{3}}$

$= \frac{-2}{1 - \frac{\sqrt{3}}{2}}$ ***A1***

$= \frac{-4}{2 - \sqrt{3}}$

$= \frac{-4}{2 - \sqrt{3}}\left(\frac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ ***M1***

$= \frac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ ***A1***

continued…

Question 11 continued

METHOD 2

$$\frac{dy}{dx} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$$ ***M1A1***

$$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$$ ***A1***

$$= \frac{-2\sec^2 \frac{\pi}{6}}{\left(\tan \frac{\pi}{6} - 1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}} - 1\right)^2} = \frac{-8}{\left(1 - \sqrt{3}\right)^2}$$ ***M1***

Note: Award ***M1*** for substitution of $\frac{\pi}{6}$.

$$\frac{-8}{\left(1 - \sqrt{3}\right)^2} = \frac{-8}{\left(4 - 2\sqrt{3}\right)} \frac{\left(4 + 2\sqrt{3}\right)}{\left(4 + 2\sqrt{3}\right)} = -8 - 4\sqrt{3}$$ ***M1A1***

[6 marks]

continued…

Question 11 continued

(d) Area $= \left|\int_0^{\frac{\pi}{6}} \frac{\sin x + \cos x}{\sin x - \cos x}\,\mathrm{d}x\right|$ ***M1***

$= \left|\left[\ln\left|\sin x - \cos x\right|\right]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln\left|\sin 0 - \cos 0\right|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3}-1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3}-1}{2}\right) = \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$ ***M1***

$= \ln\left(\sqrt{3}+1\right)$ ***AG***

[6 marks]

Total [16 marks]

12. (a) (i)–(iii) given the three roots α, β, γ, we have

$x^3 + px^2 + qx + c = (x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$= \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)(x-\gamma)$ ***A1***

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha+\beta+\gamma)$ ***AG***

$q = (\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$c = -\alpha\beta\gamma$ ***AG***

[3 marks]

(b) **METHOD 1**

i) Given $-\alpha-\beta-\gamma = -6$

And $\alpha\beta+\beta\gamma+\gamma\alpha = 18$

Let the three roots be α, β, γ.

So $\beta - \alpha = \gamma - \beta$ ***M1***

or $2\beta = \alpha + \gamma$

Attempt to solve simultaneous equations: ***M1***

$\beta + 2\beta = 6$ ***A1***

$\beta = 2$ ***AG***

ii) $\alpha + \gamma = 4$

$2\alpha + 2\gamma + \alpha\gamma = 18$

$\Rightarrow \gamma^2 - 4\gamma + 10 = 0$

$\Rightarrow \gamma = \dfrac{4 \pm \mathrm{i}\sqrt{24}}{2}$ ***(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\dfrac{4+\mathrm{i}\sqrt{24}}{2}\right)\left(\dfrac{4-\mathrm{i}\sqrt{24}}{2}\right)2 = -20$ ***A1***

[5 marks]

continued…

Question 12 continued

METHOD 2

(i) let the three roots be α, $\alpha - d$, $\alpha + d$ ***M1***
adding roots ***M1***
to give $3\alpha = 6$ ***A1***
$\alpha = 2$ ***AG***

(ii) α is a root, so $2^3 - 6 \times 2^2 + 18 \times 2 + c = 0$ ***M1***

$8 - 24 + 36 + c = 0$
$c = -20$ ***A1***

[5 marks]

METHOD 3

(i) let the three roots be α, $\alpha - d$, $\alpha + d$ ***M1***
adding roots ***M1***
to give $3\alpha = 6$ ***A1***
$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2-d) + (2-d)(2+d) + 2(2+d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}\,\mathrm{i}$
$\Rightarrow c = -20$ ***A1***

[5 marks]

continued…

Question 12 continued

(c) **METHOD 1**

Given $-\alpha-\beta-\gamma=-6$
And $\alpha\beta+\beta\gamma+\gamma\alpha=18$

Let the three roots be α,β,γ.

So $\frac{\beta}{\alpha}=\frac{\gamma}{\beta}$ ***M1***

or $\beta^2=\alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta+\gamma\beta+\beta^2=18$
$\beta(\alpha+\beta+\gamma)=18$
$6\beta=18$
$\beta=3$ ***A1***

$\alpha+\gamma=3$, $\alpha=\frac{9}{\gamma}$
$\Rightarrow\gamma^2-3\gamma+9=0$
$\Rightarrow\gamma=\frac{3\pm i\sqrt{27}}{2}$ ***(A1)(A1)***

Therefore $c=-\alpha\beta\gamma=-\left(\frac{3+i\sqrt{27}}{2}\right)\left(\frac{3-i\sqrt{27}}{2}\right)3=-27$ ***A1***

[6 marks]

METHOD 2

let the three roots be a, ar, ar^2 ***M1***
attempt at substitution of a, ar, ar^2 and *p* and *q* into equations from (a) ***M1***
$6=a+ar+ar^2\left(=a\left(1+r+r^2\right)\right)$ ***A1***
$18=a^2r+a^2r^3+a^2r^2\left(=a^2r\left(1+r+r^2\right)\right)$ ***A1***

therefore $3=ar$ ***A1***
therefore $c=-a^3r^3=-3^3=-27$ ***A1***
[6 marks]

Total [14 marks]

13. (a) $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ ***M1***

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ ***A1***

$= \sqrt{n+1}-\sqrt{n}$ ***AG***

[2 marks]

(b) $\sqrt{2}-1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ ***A2***

$< \frac{1}{\sqrt{2}}$ ***AG***

[2 marks]

(c) consider the case $n = 2$: required to prove that $1+\frac{1}{\sqrt{2}} > \sqrt{2}$ ***M1***

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2}-1$

hence $1+\frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ ***A1***

now assume true for $n = k$: $\sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ ***M1***

$\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k+1$: $\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ (***M1***)

from assumption, we have that $\frac{1}{\sqrt{1}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > \sqrt{k}+\frac{1}{\sqrt{k+1}}$ ***M1***

so attempt to show that $\sqrt{k}+\frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

continued…

Question 13 continued

EITHER

$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ ***A1***

$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k} + \sqrt{k+1}}$, (from part a), which is true ***A1***

OR

$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k+1}\sqrt{k}+1}{\sqrt{k+1}}$. ***A1***

$> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \sqrt{k+1}$ ***A1***

THEN

so true for $n = 2$ and $n = k$ true $\Rightarrow n = k+1$ true. Hence true for all $n \geq 2$ ***R1***

Note: Award ***R1*** only if all previous ***M*** marks have been awarded.

[9 marks]

Total [13 marks]